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Spontaneous Breaking of Non-Relativistic Scale Symmetry

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ABSTRACT: We analyze the mechanism of spontaneous symmetry breaking of scale invariance in Galilean invariant field theories. We show that there is no dynamical gapless dilaton mode unless the $U(1)$ particle number symmetry is spontaneously broken. When both scale and particle number symmetries are spontaneously broken there is one propagating gapless Nambu-Goldstone mode. Its dispersion relation is linear if the chemical potential is nonzero and quadratic otherwise. We discuss the reversibility of RG flows in such theories.

KEYWORDS: Spontaneous Symmetry Breaking

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1 Introduction

Spontaneous symmetry breaking (SSB) underlies a large number of physical phenomena such as superconductivity, superfluidity and the generation of elementary particle masses. A spontaneous breaking of a continuous global symmetry implies via the Nambu-Goldstone (NG) theorem the existence of a gapless Nambu-Goldstone mode. In a relativistic field theory there is one NG mode for each broken symmetry generator [1–3]. This one to one correspondence between broken generators and gapless NG modes does not hold when spacetime symmetries are spontaneously broken [4, 5]. It is also not generally the case for spontaneously broken global symmetries in non-relativistic field theories [6–8].

When scale symmetry is spontaneously broken in a relativistic field theory, there is a corresponding NG mode, the dilaton. The dilaton effective action encodes, for instance, the information about the A-type conformal anomaly and has been valuable in proving the a-theorem in [9]. This leads naturally to inquire about the mechanism of spontaneously broken scale symmetry in non-relativistic field theories. Such field theories have much importance in the study of low energy condensed matter systems, as well as in non-relativistic holography (e.g. [10]).

The aim of this paper is to analyze SSB of scale invariance in Galilean invariant field theories. In non-relativistic field theories space and time scale differently: $\vec{x} \rightarrow e^\sigma \vec{x}, t \rightarrow e^{z\sigma} t$, where σ is a real parameter and \vec{x} is a d -dimensional vector. z is called a dynamical exponent and we will consider the Galilean case $z = 2$. We will show that there is no gapless dilaton mode in a Galilean field theory unless the $U(1)$ particle number symmetry is also spontaneously broken. Particle number symmetry is an additional symmetry of Galilean field theories that does not exist in the relativistic case. We will see that when both scale

and particle number symmetries are spontaneously broken there is just one propagating gapless mode.

In the Galilean algebra the generator M of the $U(1)$ particle number symmetry is a central extension. It appears in the commutator of translations P_i and Galilean boosts K_j

$$[P_i, K_j] = -i\delta_{ij}M, \quad i = 1, \dots, d. \quad (1.1)$$

The fact that having a dynamical NG boson for scale symmetry breaking requires a spontaneous breaking of the $U(1)$ particle symmetry may be anticipated, since otherwise the vacuum is empty. This is in contrast to relativistic field theory vacuum where pair production exists.

The broken symmetry generators create NG modes from the vacuum and their commutator algebra may impose relations between these modes. Such an example is the relation among the NG mode related to SSB of the $U(1)$ particle number symmetry and the mode related to the SSB of boosts. Denote by θ the NG mode related to the $U(1)$ symmetry and by β_i the NG modes related to boosts. From the commutator (1.1), one gets when translation is not broken that $\beta_i \sim \partial_i \theta$. Thus, while this commutator predicts that $U(1)$ SSB implies that Galilean boosts are spontaneously broken, it also predicts that there is only one independent NG mode. This relationship is an example of a general structure called Inverse Higgs Constraints (IHC) [11–13]. Such an argument, however, does not explain why when scale and $U(1)$ are spontaneously broken there is only one NG mode. We will derive this result by an explicit calculation of the spectrum.

Based on $z = 2$ dimensional arguments we may anticipate the dispersion relation of the NG mode. If we denote by v the symmetry breaking length scale, then the dispersion relation takes the general form

$$\omega = \frac{\vec{k}^2}{2m} F(kv). \quad (1.2)$$

We'll see that this is indeed the behavior where with zero chemical potential $F(0)$ is finite and we get to first order a quadratic dispersion relation and with a nonzero chemical potential we get to first order a linear dispersion relation instead of a quadratic one.

There are various methods to construct the effective action of NG bosons. One way is to write all possible terms that respect all the symmetries. Another way is to couple the theory to curved external background sources. We will consider both methods in the study of the non-relativistic dilaton.

The paper is organized as follows. In the next section we will begin by considering the NG effective action based on symmetry arguments. We will see that spontaneously breaking scale invariance while maintaining the $U(1)$ symmetry does not allow for a propagating dilaton mode. We will then spontaneously break the $U(1)$ symmetry as well. We will construct the NG effective action at leading orders in the derivatives expansion. Next, we will derive the same effective action using a coupling to the Newton-Cartan curved geometry. We will analyze the spectrum of the NG bosons and find one gapless propagating mode. Finally, we will discuss the possible relevance of the results to RG flow theorems in Galilean field theories and conclude with a brief summary and outlook.

2 The NG Boson Effective Action

2.1 Symmetries

In this subsection we derive the first few terms in a derivative expansion of the effective action of the non-relativistic NG boson following a spontaneous breaking of scale invariance $(\vec{x}, t) \rightarrow (e^\sigma \vec{x}, e^{z\sigma} t)$, using the symmetries of the theory.

2.1.1 Unbroken $U(1)$ Particle Number Symmetry

The dilaton is a real field which we will denote by τ . It carries zero $U(1)$ charge. Under scale transformation the dilaton transforms as

$$\tau(\vec{x}, t) \rightarrow \tau(e^\sigma \vec{x}, e^{z\sigma} t) + \sigma. \quad (2.1)$$

Since $dt d^d x \rightarrow e^{-(z+d)\sigma} dt d^d x$, $\partial_t \rightarrow e^{z\sigma} \partial_t$ and $\partial_i \rightarrow e^\sigma \partial_i$, the effective Lagrangian density should take the form

$$L = \mathcal{L}(e^{-z\tau} \partial_t, e^{-\tau} \partial_i) e^{(z+d)\tau}. \quad (2.2)$$

where \mathcal{L} is a general polynomial in these differential operators.

Under $z = 2$ non-relativistic boost transformation with boost parameters \vec{u} , the dilaton and its derivatives transform as

$$\begin{aligned} \tau(\vec{x}, t) &\rightarrow \tau(\vec{x} - \vec{u}t, t), \\ \partial_i \tau(\vec{x}, t) &\rightarrow \partial_i \tau(\vec{x} - \vec{u}t, t), \\ \partial_t \tau(\vec{x}, t) &\rightarrow \partial_t \tau(\vec{x} - \vec{u}t, t) - u_i \partial_i \tau(\vec{x} - \vec{u}t, t). \end{aligned} \quad (2.3)$$

The ∂_i term is invariant while the ∂_t term is not. This basically forbids any time derivative term in the effective Lagrangian, hence there is no dynamical dilaton.

Consider this more explicitly. Zeroth order derivative terms can't by themselves generate dynamics, however, they are allowed provided the scale transformation is fixed correctly as follows:

$$L_0 = \Lambda e^{(z+d)\tau}, \quad (2.4)$$

where Λ is a dimensionless constant. First order derivatives terms are total derivatives, so next we consider the second order derivatives terms. By rotational symmetry, the only possible two derivatives terms are $(\partial_t \tau)^2$ and $(\partial_i \tau)^2$. Thus, up to two derivatives the effective dilaton Lagrangian should read

$$L = e^{(-z+d)\tau} (\partial_t \tau)^2 - \gamma e^{(z+d-2)\tau} (\partial_i \tau)^2 + \Lambda e^{(z+d)\tau}. \quad (2.5)$$

The first term is not invariant under $z = 2$ non-relativistic boost transformation. One can repeat the same analysis at higher order in derivatives arriving to the same conclusion that there is no dynamical dilaton. Note, that if one works in Lifshitz field theory but without imposing Galilean boost invariance, one can have a dynamical dilaton. For example, the $z = 2$ scale invariant terms $e^{(d-2)\tau} (\partial_t \tau)^2$, $e^{(d-2)\tau} (\partial_i^2 \tau)^2$, $e^{(d-2)\tau} (\partial_i \tau)^4$, $e^{(d-2)\tau} (\partial_i \tau)^2 (\partial_i^2 \tau)$, combine using $\phi = e^{\frac{(d-2)}{2}\tau}$ to give

$$L = \frac{1}{2} (\partial_t \phi)^2 - \frac{\kappa}{4} (\partial_i^2 \phi)^2. \quad (2.6)$$

This is the Lagrangian for a free Lifshitz real scalar, which is not boost invariant.

2.1.2 Broken $U(1)$ Particle Number Symmetry

In the following both scale and $U(1)$ particle number are spontaneously broken and we will consider a complex scalar field $\phi = e^{\Delta\tau + im\theta}$ that carries a $U(1)$ charge m and scaling dimension Δ (where τ corresponds to the dilaton and θ to the $U(1)$ NG mode). Under $z = 2$ boost transformation one has

$$\phi(\vec{x}, t) \rightarrow \phi(\vec{x} - \vec{u}t, t) e^{-\frac{i}{2}m\vec{u}^2 t + im\vec{u} \cdot \vec{x}}, \quad (2.7)$$

or τ invariant and

$$\theta \rightarrow \theta - \frac{1}{2}\vec{u}^2 t + \vec{u} \cdot \vec{x}. \quad (2.8)$$

The zero derivatives term $\phi\phi^*$ is invariant and gives (2.4) with $\Delta = \frac{d+2}{2}$. At leading order in derivatives we can write a scale, rotation and boost invariant Lagrangian

$$L_1 = \frac{i}{2}(\phi^* \partial_t \phi - \partial_t \phi^* \phi) - \frac{1}{2m} \partial_i \phi \partial_i \phi^*, \quad (2.9)$$

with $\Delta = \frac{d}{2}$.

More generally, we can build a boost invariant structure (up to a phase) $-i\partial_t \phi - \frac{1}{2m} \partial_i^2 \phi$. It can be used for the construction of a boost and scale invariant Lagrangian, so to next order in derivatives one has also

$$L_2 = \left(i\partial_t \phi^* - \frac{1}{2m} \partial_i^2 \phi^* \right) \left(-i\partial_t \phi - \frac{1}{2m} \partial_i^2 \phi \right) e^{2(\Delta' - \Delta)\tau}, \quad (2.10)$$

where ϕ has $U(1)$ charge m and scaling dimension Δ , and $\Delta' = \frac{d}{2} - 1$. In terms of τ and θ

$$L_2 = \left| i\partial_t (\Delta\tau - im\theta) - \frac{1}{2m} (\partial_i (\Delta\tau - im\theta))^2 - \frac{1}{2m} \partial_i^2 (\Delta\tau - im\theta) \right|^2 e^{2\Delta'\tau}. \quad (2.11)$$

We can also consider the case of theories which are invariant under the full Schrödinger group, which contains in addition the special conformal transformation, given by:

$$\phi(\vec{x}, t) \rightarrow (1 + \nu t)^{-\Delta} e^{i\frac{m}{2} \frac{\nu x^2}{1 + \nu t}} \phi\left(\frac{\vec{x}}{1 + \nu t}, \frac{t}{1 + \nu t}\right), \quad (2.12)$$

or in terms of τ and θ :

$$\begin{aligned} \tau(\vec{x}, t) &\rightarrow \tau\left(\frac{\vec{x}}{1 + \nu t}, \frac{t}{1 + \nu t}\right) - \ln(1 + \nu t), \\ \theta(\vec{x}, t) &\rightarrow \theta\left(\frac{\vec{x}}{1 + \nu t}, \frac{t}{1 + \nu t}\right) + \frac{\nu x^2}{2(1 + \nu t)}. \end{aligned} \quad (2.13)$$

It is easily verified that the previously mentioned structure $-i\partial_t \phi - \frac{1}{2m} \partial_i^2 \phi$ is covariant under these special conformal transformations (that is, invariant up to a phase and a scale factor), only when $\Delta = \frac{d}{2}$ (or when $m = 0$, which corresponds to terms that depend only on τ and its spatial derivatives). Therefore the Lagrangians (2.4) (with $\Delta = \frac{d+2}{2}$) and (2.9) (with $\Delta = \frac{d}{2}$) are indeed Schrödinger invariant. However, the requirement for Schrödinger invariance restricts the higher order Lagrangian (2.10) to the case of $\Delta = \frac{d}{2}$ (or alternatively $m = 0$).

2.2 Geometry

One can construct the NG boson effective action by geometrical considerations: coupling the field theory to a curved background, promoting the symmetries to local ones, looking at all possible actions in this framework and taking the flat background limit. This was done for the relativistic case [14] where the curved background is a Riemannian geometry with a metric tensor $G_{\mu\nu}$. One constructs a Weyl invariant metric $\hat{G}_{\mu\nu} = e^{-2\tau} G_{\mu\nu}$ and writes the effective action in terms of scalar terms constructed from it.

In the non-relativistic case one has to use the Newton-Cartan (NC) geometry instead (see e.g. [15, 17]). The NC geometry is built from a time direction described by a 1-form n_μ , a spatial metric $h^{\mu\nu}$ orthogonal to n_μ and a $U(1)$ gauge field A_μ which couples to the conserved particle number current. Further, one defines a vector v^μ that satisfies $v^\mu n_\mu = 1$ and induces a metric $h_{\mu\nu}$ satisfying

$$h_{\mu\nu} v^\nu = 0, \quad h_{\mu\rho} h^{\nu\rho} = P_\mu^\nu = \delta_\mu^\nu - v^\nu n_\mu. \quad (2.14)$$

These definitions are not unique because we can redefine v^μ using an arbitrary vector ψ_ν

$$v^\mu \rightarrow v^\mu + h^{\mu\nu} \psi_\nu, \quad (2.15)$$

and redefine $h_{\mu\nu}$ correspondingly:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - (n_\mu P_\nu^\rho + n_\nu P_\mu^\rho) \psi_\rho + n_\mu n_\nu h^{\rho\sigma} \psi_\rho \psi_\sigma, \quad (2.16)$$

together with the following redefinition of A_μ :

$$A_\mu \rightarrow A_\mu + P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu h^{\nu\rho} \psi_\nu \psi_\rho. \quad (2.17)$$

These transformations are the Milne boosts and one can define Milne boost invariant objects, as follows:

$$v_A^\mu = v^\mu - h^{\mu\nu} A_\nu, \quad g_{\mu\nu} = (h_A)_{\mu\nu} = h_{\mu\nu} + n_\mu A_\nu + n_\nu A_\mu. \quad (2.18)$$

Since we are interested in the ($z = 2$ anisotropic-)Weyl (scale) and $U(1)$ symmetries, we need the Weyl and $U(1)$ transformations of the NC objects. The $U(1)$ gauge symmetry transforms only A_μ out of the basic structures:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (2.19)$$

while the Weyl symmetry transforms n_μ and $h_{\mu\nu}$:

$$h_{\mu\nu} \rightarrow e^{2\sigma} h_{\mu\nu}, \quad n_\mu \rightarrow e^{2\sigma} n_\mu. \quad (2.20)$$

There are two equivalent ways to construct the NG boson effective action for the scale and $U(1)$ spontaneous breaking. One is to introduce spectator fields, τ for the scale symmetry and θ for the $U(1)$ symmetry, with τ transforming under scale like a dilaton $\tau \rightarrow \tau + \sigma$, and θ transforming under $U(1)$ as $\theta \rightarrow \theta + \Lambda$ where $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ (and both

invariant under Milne boosts). Using these spectator fields and the NC structures h, n, A , we can then define Weyl and gauge invariant geometric quantities $\hat{h}, \hat{n}, \hat{A}$ as follows:

$$\hat{h}_{\mu\nu} \equiv e^{-2\tau} h_{\mu\nu}, \quad \hat{n}_\mu \equiv e^{-2\tau} n_\mu, \quad \hat{A}_\mu \equiv A_\mu - \partial_\mu \theta. \quad (2.21)$$

From these structures, we can build boost invariant scalars. This procedure is made straightforward by defining the boost invariant metric \hat{g} and vector \hat{v}_A^μ as follows:

$$\begin{aligned} \hat{v}_A^\mu &\equiv e^{2\tau} (v^\mu - h^{\mu\nu} (A_\nu - \partial_\nu \theta)), \\ \hat{g}_{\mu\nu} &\equiv e^{-2\tau} (h_{\mu\nu} + n_\mu (A_\nu - \partial_\nu \theta) + n_\nu (A_\mu - \partial_\mu \theta)). \end{aligned} \quad (2.22)$$

Finally, we take the limit where the geometry is flat:

$$h^{\mu\nu} \partial_\mu \partial_\nu = \delta^{ij} \partial_i \partial_j, \quad A = 0, \quad n_\mu dx^\mu = dt, \quad v^\mu \partial_\mu = \partial_t, \quad (2.23)$$

and thus remain only with the spectator fields. This proposal is a generalization of the relativistic case discussed above, where the dilaton factor was added to the metric $G_{\mu\nu}$ to compensate for the Weyl variation. In our case we have the boost invariant structures $g_{\mu\nu}$ and v_A^μ given in (2.18), as well as n_μ and $h^{\mu\nu}$. Here again, to enforce Weyl invariance we add τ , and to enforce $U(1)$ invariance we add θ . The structures $\hat{g}_{\mu\nu}, \hat{v}_A^\mu, \hat{n}_\mu, \hat{h}^{\mu\nu}$ are therefore Milne boost invariant (by the construction of $g_{\mu\nu}, v_A^\mu, n_\mu, h^{\mu\nu}$), Weyl invariant (by the use of the compensator τ), and $U(1)$ invariant (by the use of the compensator θ).

The other equivalent way to construct the effective action, which is the one we will pursue, is to take Milne boost invariants and perform Weyl and $U(1)$ transformations with parameters τ and θ , respectively. This will evidently give the same answer (up to the sign of the fields τ and θ , which will be opposite to the one in the first method, and therefore to the convention used in [9]), since the Weyl and $U(1)$ transformations will force the appearance of τ and θ in exactly the right form such that if they themselves transform under Weyl and $U(1)$, the whole expression would be invariant.

Using the boost invariant scalars, we will get the effective action for τ and θ after restricting to flat geometry (2.23). Note, that the flat geometry restriction is respected by a combination of a Milne boost and a $U(1)$ transformation. Under this combination, θ transforms in the same way as it transformed under non-relativistic boost transformations in the previous section (2.8). The reason is that under the appropriate Milne boost transformation parametrized by $\psi_\mu = (0, \vec{u})$ at flat geometry, A_μ transforms as in (2.17), which in this case takes the form $A_\mu \rightarrow A_\mu + \partial_\mu (-\frac{1}{2}\vec{u}^2 t + \vec{u} \cdot \vec{x})$. In order to compensate for that transformation, the $U(1)$ transformation should be (again, at flat geometry) $\Lambda = \frac{1}{2}\vec{u}^2 t - \vec{u} \cdot \vec{x}$, which produces the correct boost transformation of θ . Also note that the flat geometry restriction is respected by an additional combination of Milne boosts, $U(1)$ transformations and anisotropic Weyl transformations, which corresponds to the special conformal transformation (2.13) (see [15]). Therefore the effective action obtained using this geometric method necessarily corresponds to the full Schrödinger invariant case.

We wish to list the invariant scalars to leading order in derivatives. Note that derivative counting should be done after restricting to flat geometry. Consider the Milne boost

invariant metric and vector (2.18). The simplest geometrical term is the cosmological constant term (constant up to the $\sqrt{\det(\gamma_{\mu\nu})}$ factor, where $\gamma_{\mu\nu} \equiv h_{\mu\nu} + n_\mu n_\nu$, that contributes the τ dependence and ensures Weyl invariance), which matches the non-dynamical term $e^{(z+d)\tau}$ (2.4) discussed previously.

The next simplest scalar one can build is $g_{\mu\nu} v_A^\mu v_A^\nu$. We perform Weyl transformation and $U(1)$ transformation and then restrict to flat geometry to get an expression in terms of τ and θ . Counting derivatives naively in this expression, g may contribute one derivative and v_A may contribute one space derivative, so we might be lead to think that in total we have 3 derivatives. However, since $h^{\mu\nu} n_\mu = 0$, the full expression, when restricted to flat geometry, has only one time derivative or 2 space derivatives. Written explicitly, we get:

$$g_{\mu\nu} v_A^\mu v_A^\nu \rightarrow e^{-2\tau} \left(2\partial_t \theta - (\partial_i \theta)^2 \right). \quad (2.24)$$

The other term at this order in derivatives is the spatial Ricci scalar \tilde{R} corresponding to the standard Levi-Civita connection of the metric induced on the space foliation.¹ Since the metric induced on the foliation is invariant under Milne boosts (i.e. $g_{\mu\nu} u^\mu w^\nu$ is boost invariant for any space tangent vectors u^μ, w^μ), \tilde{R} is boost invariant as well. It is also gauge invariant, and therefore it depends only on τ . By Weyl transformation we get

$$\tilde{R} \rightarrow e^{-2\tau} \left(-2(d-1)\partial_i^2 \tau - d(d-1)(\partial_i \tau)^2 \right). \quad (2.25)$$

There are more terms, e.g. $a_\mu a^\mu$ where $a_\mu \equiv -\mathcal{L}_v n_\mu$ (see e.g. [17]), but when we restrict to conformally flat background and consider integration by parts, they are all equivalent to the terms above. Therefore, these two expressions complete the list of terms up to one time derivative or two space derivatives (i.e. the same order in $z = 2$ Lifshitz scaling counting). From the last two expressions, (2.24) and (2.25), we construct the leading order Lagrangian for the complex field ϕ (2.9).

For higher orders in derivatives, one can define a boost invariant affine connection from the structures $n_\mu, h^{\mu\nu}, v_A^\mu, g_{\mu\nu}$, as well as the corresponding Riemann tensor $R^\lambda_{\mu\sigma\nu}$ and Ricci tensor $R_{\mu\nu} \equiv R^\sigma_{\mu\sigma\nu}$ (see [15]). One can then obtain various higher derivative boost invariant scalars, such as $R_{\mu\nu} h^{\mu\nu}$ and $R_{\mu\nu} v_A^\mu v_A^\nu$, as well as purely spatial ones such as \tilde{R}^2 and a^4 . As before, by performing Weyl and gauge transformations on a linear combination of these scalars and restricting to flat geometry, one can construct the higher order Lagrangian (2.10) (as well as other higher derivatives terms, which do not affect the conclusions of the analysis in the next subsection).

2.3 Spectrum Analysis

In the following we will analyze the spectrum of the low energy theory of τ and θ . We use the notation $\phi_{\Delta,m} = e^{\Delta\tau + im\theta}$ for a complex scalar field of dimension Δ and mass m , and ϕ_k as a shorthand for ϕ_{Δ_k, m_k} . The leading order boost, $U(1)$ and scale invariant Lagrangian

¹Since in this context we are considering NC geometries which are conformally flat, i.e. $n = e^{2\tau} dt$, we can safely assume that n_μ satisfies the Frobenius condition and therefore induces a foliation of the spacetime manifold into equal time slices. See [17] for further discussion.

which we derived in the previous sections reads

$$L = \Lambda \phi_0^* \phi_0 + A \left[\frac{i}{2} (\phi_1^* \partial_t \phi_1 - \partial_t \phi_1^* \phi_1) - \frac{1}{2m_1} \partial_i \phi_1^* \partial_i \phi_1 \right] + B \left[\left(i \partial_t \phi_2^* - \frac{1}{2m_2} \partial_i^2 \phi_2^* \right) \left(-i \partial_t \phi_2 - \frac{1}{2m_2} \partial_i^2 \phi_2 \right) \right], \quad (2.26)$$

where the dimensions Δ_i are fixed by scale invariance to the following values:

$$\Delta_0 = \frac{d+2}{2}, \quad \Delta_1 = \frac{d}{2}, \quad \Delta_2 = \frac{d-2}{2}. \quad (2.27)$$

Note that this is not the most general expression one could build up to this order in derivatives – we could take a linear combination of terms of this form using fields $\phi_{\Delta,m}$ with various values of Δ and m , while compensating for the dimension by multiplying each by an appropriate exponent of τ (as in (2.10)). However, for the cosmological constant term (the Λ term) and the leading term in derivatives (the A term), one can always rewrite these terms in the form given in (2.26) (using just ϕ_0 and ϕ_1 with no extra exponents of τ), with an appropriate choice of the parameters Λ , A and m_1 . For the subleading term in derivatives (the B term), this is not true in general, but since we will be mainly interested in the leading contributions of the Λ and A terms, we will assume the form given in (2.26) as an example for the contribution of subleading terms in derivatives. We therefore use the Lagrangian (2.26), for which all τ dependence is through the ϕ_0 , ϕ_1 and ϕ_2 fields. It is also important to note that the subleading term in (2.26) is not Schrödinger invariant, as mentioned in subsection 2.1.2. Full Schrödinger invariance restricts this subleading term to the form (2.10) with $\Delta_2 = \frac{d}{2}$. However, using such a term instead in (2.26) does not change the main results of this subsection.

We derive the equations of motion by varying the action defined by (2.26) with respect to τ and θ , through ϕ_0 , ϕ_1 and ϕ_2 . We have $\delta S = \frac{\delta S}{\delta \phi_i} \delta \phi_i + c.c.$, where:

$$\begin{aligned} \frac{1}{\Lambda} \frac{\delta S}{\delta \phi_0} &= \phi_0^*, \\ \frac{1}{A} \frac{\delta S}{\delta \phi_1} &= -i \partial_t \phi_1^* + \frac{1}{2m_1} \partial_i^2 \phi_1^*, \\ \frac{1}{B} \frac{\delta S}{\delta \phi_2} &= -\partial_t^2 \phi_2^* - \frac{i}{m_2} \partial_t \partial_i^2 \phi_2^* + \frac{1}{4m_2^2} \partial_i^2 \partial_i^2 \phi_2^*, \end{aligned} \quad (2.28)$$

and their complex conjugates.

In terms of τ and θ we have $\delta \phi_{\Delta,m} = (\Delta \delta \tau + im \delta \theta) \phi_{\Delta,m}$, and therefore:

$$\begin{aligned} \delta S &= \frac{\delta S}{\delta \phi_0} \phi_0 (\Delta_0 \delta \tau + im_0 \delta \theta) + \frac{\delta S}{\delta \phi_1} \phi_1 (\Delta_1 \delta \tau + im_1 \delta \theta) \\ &\quad + \frac{\delta S}{\delta \phi_2} \phi_2 (\Delta_2 \delta \tau + im_2 \delta \theta) + c.c. . \end{aligned} \quad (2.29)$$

Separating the variation by τ and θ we have

$$\begin{aligned} \frac{\delta S}{\delta \tau} &\propto \Delta_0 \Re \frac{\delta S}{\delta \phi_0} \phi_0 + \Delta_1 \Re \frac{\delta S}{\delta \phi_1} \phi_1 + \Delta_2 \Re \frac{\delta S}{\delta \phi_2} \phi_2, \\ \frac{\delta S}{\delta \theta} &\propto m_1 \Im \frac{\delta S}{\delta \phi_1} \phi_1 + m_2 \Im \frac{\delta S}{\delta \phi_2} \phi_2, \end{aligned} \quad (2.30)$$

where \Re and \Im denote the real and imaginary parts respectively.

We will perturb the equations of motion around fixed values

$$\tau = \hat{\tau}, \quad \partial_t \theta = -\hat{\mu}, \quad (2.31)$$

where $\hat{\mu}$ is the chemical potential.² The particular value of $\hat{\tau}$ can be absorbed by a redefinition of A, B and Λ , so we will take it to be zero. Note however that $v \equiv e^{-\hat{\tau}}$ quantifies the scale of the state we are perturbing about.

We express $\frac{\delta S}{\delta \phi_1} \phi_1$ and $\frac{\delta S}{\delta \phi_2} \phi_2$ using τ and θ (see (2.28)):

$$\begin{aligned} \frac{1}{A} \frac{\delta S}{\delta \phi_1} \phi_1 &= -i (\partial_t \phi_1^*) \phi_1 + \frac{1}{2m_1} (\partial_i^2 \phi_1^*) \phi_1 \\ &= e^{2\Delta_1 \tau} (-i \Delta_1 \partial_t \tau - m_1 \partial_t \theta) + \\ &\quad + \frac{e^{2\Delta_1 \tau}}{2m_1} \left(\Delta_1 \partial_i^2 \tau - i m_1 \partial_i^2 \theta + \Delta_1^2 (\partial_i \tau)^2 - m_1^2 (\partial_i \theta)^2 - 2i m_1 \Delta_1 \partial_i \tau \partial_i \theta \right) \\ &\simeq e^{2\Delta_1 \tau} \left(-i \Delta_1 \partial_t \tau - m_1 \partial_t \theta + \frac{1}{2m_1} (\Delta_1 \partial_i^2 \tau - i m_1 \partial_i^2 \theta) \right), \end{aligned} \quad (2.32)$$

where \simeq here denotes keeping only terms that contribute up to linear order in the τ and θ perturbation. Similarly, again up to linear order, we have

$$\begin{aligned} \frac{1}{B} \frac{\delta S}{\delta \phi_2} \phi_2 &= -(\partial_t^2 \phi_2^*) \phi_2 - \frac{i}{m_2} (\partial_t \partial_i^2 \phi_2^*) \phi_2 + \frac{1}{4m_2^2} (\partial_i^2 \partial_i^2 \phi_2^*) \phi_2 = \\ &\simeq e^{2\Delta_2 \tau} \left(-\Delta_2 \partial_t^2 \tau + i m_2 \partial_t^2 \theta + m_2^2 (\partial_t \theta)^2 + 2i \Delta_2 m_2 \partial_t \theta \partial_t \tau \right) - \\ &\quad - \frac{i}{m_2} e^{2\Delta_2 \tau} (\Delta_2 \partial_t \partial_i^2 \tau - i m_2 \partial_t \partial_i^2 \theta - m_2^2 \partial_i^2 \theta \partial_t \theta - i \Delta_2 m_2 \partial_i^2 \tau \partial_t \theta) + \\ &\quad + \frac{1}{4m_2^2} e^{2\Delta_2 \tau} (\Delta_2 \partial_i^2 \partial_i^2 \tau - i m_2 \partial_i^2 \partial_i^2 \theta). \end{aligned} \quad (2.33)$$

Combining the last two equations, the variation of the action with respect to τ and θ is given by the following expressions up to linear order in the perturbation:

$$\begin{aligned} \frac{\delta S}{\delta \tau} &\propto \Lambda \Delta_0 e^{2\Delta_0 \tau} + A \Delta_1 e^{2\Delta_1 \tau} \left(-m_1 \partial_t \theta + \frac{\Delta_1}{2m_1} \partial_i^2 \tau \right) \\ &\quad + B \Delta_2 e^{2\Delta_2 \tau} \left(-\Delta_2 \partial_t^2 \tau + m_2^2 (\partial_t \theta)^2 - \partial_t \partial_i^2 \theta - \Delta_2 \partial_i^2 \tau \partial_t \theta + \frac{\Delta_2}{4m_2^2} \partial_i^2 \partial_i^2 \tau \right), \\ \frac{\delta S}{\delta \theta} &\propto A m_1 e^{2\Delta_1 \tau} \left(-\Delta_1 \partial_t \tau - \frac{1}{2} \partial_i^2 \theta \right) \\ &\quad + B m_2 e^{2\Delta_2 \tau} \left(m_2 \partial_t^2 \theta + 2 \Delta_2 m_2 \partial_t \theta \partial_t \tau - \frac{\Delta_2}{m_2} \partial_t \partial_i^2 \tau + m_2 \partial_i^2 \theta \partial_t \theta - \frac{1}{4m_2} \partial_i^2 \partial_i^2 \theta \right). \end{aligned} \quad (2.34)$$

The zeroth order equation requires

$$\Lambda \Delta_0 = -A m_1 \Delta_1 \hat{\mu} - B m_2^2 \Delta_2 \hat{\mu}^2, \quad (2.35)$$

²Note that our sign convention for θ is different from that in [16].

and we see that the static potential value (cosmological constant) Λ corresponds to a non-zero chemical potential $\hat{\mu}$ (although a non-zero chemical potential is possible even when Λ vanishes).

Consider first the case of a non-zero chemical potential $\hat{\mu}$. We get the following linearized equations of motion to leading order in momentum \vec{k} and energy ω ,

$$\begin{pmatrix} -A\Delta_1 m_1 \omega - 2B\Delta_2 m_2^2 \hat{\mu} \omega & 2\Lambda\Delta_0^2 + 2A\Delta_1^2 m_1 \hat{\mu} + 2B\Delta_2^2 m_2^2 \hat{\mu}^2 \\ \frac{1}{2}A m_1 \vec{k}^2 - B m_2^2 \omega^2 + B m_2^2 \hat{\mu} \vec{k}^2 & A\Delta_1 m_1 \omega + 2B\Delta_2 m_2^2 \hat{\mu} \omega \end{pmatrix} \begin{pmatrix} i\delta\tilde{\theta} \\ \delta\tilde{\tau} \end{pmatrix} = 0, \quad (2.36)$$

where $\delta\tilde{\tau}$ denotes the Fourier transform of the τ perturbation around 0, and $\delta\tilde{\theta}$ denotes the Fourier transform of the θ perturbation around $-\hat{\mu}t$. From these equations we obtain the following dispersion relation:

$$\omega^2 = \frac{1}{\Delta_1} \hat{\mu} \vec{k}^2. \quad (2.37)$$

We can see that in this case, in the limit $k \rightarrow 0$, the dispersion relation is linear $\omega \sim k$ and $\delta\tilde{\tau} \sim \omega\delta\tilde{\theta}$. This is consistent with the analysis in [16]. Note, that since $\delta\tilde{\tau} \sim \omega\delta\tilde{\theta}$, the perturbation is mainly in θ , which may also justify ignoring the τ contribution in the leading order superfluid effective field theory as was implicitly done in [16]. Also note that the stability of the modes in the $k \rightarrow 0$ limit requires $\hat{\mu} > 0$.

The result (2.37) implies the speed of sound $v_s \equiv \sqrt{\frac{1}{\Delta_1} \hat{\mu}} = \sqrt{\frac{2}{d} \hat{\mu}}$. This can be easily understood from dimensional analysis considerations: In the presence of a $z = 2$ Lifshitz scale invariance, we can expect the relation between the chemical potential $\hat{\mu}$ and the conserved particle number (or mass) density ρ to be of the form $\hat{\mu} = C\rho^{2/d}$, where C is some dimensionless parameter. From standard thermodynamic relations, the speed of sound will be given by:

$$v_s^2 = \frac{\partial P}{\partial \rho} = \rho \frac{\partial \hat{\mu}}{\partial \rho} = \frac{2}{d} \hat{\mu}. \quad (2.38)$$

Consider next the case $\Lambda = \hat{\mu} = 0$. The linearized equations are given by:

$$\begin{pmatrix} A\Delta_1 m_1 \omega - B\Delta_2 \omega \vec{k}^2 & A\frac{\Delta_1^2}{2m_1} \vec{k}^2 - B\Delta_2^2 \omega^2 - B\frac{\Delta_2^2}{4m_2^2} \vec{k}^4 \\ \frac{1}{2}A m_1 \vec{k}^2 - B m_2^2 \omega^2 - \frac{1}{4}B \vec{k}^4 & A\Delta_1 m_1 \omega - B\Delta_2 \omega \vec{k}^2 \end{pmatrix} \begin{pmatrix} i\delta\tilde{\theta} \\ \delta\tilde{\tau} \end{pmatrix} = 0. \quad (2.39)$$

When $A \neq 0$ we have at leading order for small values of \vec{k} (and therefore ω) the non-relativistic dispersion relation

$$\omega = \frac{\vec{k}^2}{2m_1}, \quad \delta\tilde{\tau} = -i\frac{m_1}{\Delta_1} \delta\tilde{\theta}. \quad (2.40)$$

Thus, we find one gapless mode at large length scales compared to the breaking scales. The corrections can be computed to give $\omega^2 = \alpha k^4 + \beta k^6 + \dots$, where the coefficients α, β, \dots are determined from expanding the determinant of (2.39) to growing orders in \vec{k}^2 . Note that the limit $m_1 \rightarrow 0$ takes us from the broken $U(1)$ to the unbroken $U(1)$ case. The dispersion relation (2.40) blows up and we are left with no propagating mode.

In addition to the gapless mode, we have also a gapped mode as can be seen by setting $\vec{k} = 0$ in (2.39) and we get

$$\omega^2 = \frac{A^2 \Delta_1^2 m_1^2}{B^2 \Delta_2^2 m_2^2} > 0, \quad \delta\tilde{\tau} = -i \frac{m_2}{\Delta_2} \delta\tilde{\theta} . \quad (2.41)$$

Finally, note that we can obtain the case of $U(1)$ SSB without scale invariance from the above equation (2.34). Take τ to be a constant rather than a dynamical field, and to first order the variation with respect to θ gives

$$\frac{\delta S}{\delta\theta} \propto A m_1 e^{2\Delta_1 \tau} \left(-\frac{1}{2} \partial_i^2 \theta \right) + B m_2 e^{2\Delta_2 \tau} (m_2 \partial_t^2 \theta - m_2 \hat{\mu} \partial_i^2 \theta) , \quad (2.42)$$

which leads to a linear dispersion relation. Note that the cosmological constant term does not contribute to this result.

3 On a Non-Relativistic a-theorem and the Frobenius condition

The relativistic dilaton effective action was valuable for the proof of the a-theorem [9] in 3+1 dimensions, i.e. the coefficient of the A-type conformal anomaly a satisfies $a_{\text{IR}} < a_{\text{UV}}$. In the following we will make a few comments on the non-relativistic Galilean case and the feasibility of using similar arguments to prove an RG flow theorem in case such a theorem indeed holds.

In [9], the RG flow of a general relativistic field theory from a UV to an IR fixed point was studied by weakly coupling the theory to a dilaton as a conformal compensator, and matching the conformal anomalies between the UV and IR theories. In particular, the A-type anomaly of the theory contributes to the effective action of the dilaton in the IR, and therefore the S-matrix of dilaton scattering. The a-theorem then follows from unitarity of the theory.

In the case of non-relativistic field theories, invariance under a Lifshitz scale symmetry implies the following Ward identity for the stress-energy tensor:

$$D \equiv T^{\mu\nu} h_{\mu\nu} - z T^{\mu\nu} n_\mu n_\nu = 0 , \quad (3.1)$$

which is just a generalized version of the conformal tracelessness condition. However, similarly to the relativistic case, the scale symmetry can be violated due to quantum anomalies (analogous to the conformal anomalies) [17–23]. The expectation value of the stress-energy tensor on a curved spacetime manifold then no longer satisfies identity (3.1). It instead acquires an anomalous contribution:³

$$\langle D \rangle \equiv \langle T^{\mu\nu} \rangle h_{\mu\nu} - z \langle T^{\mu\nu} \rangle n_\mu n_\nu = \mathcal{A} , \quad (3.2)$$

where \mathcal{A} is a local functional of the background fields, and the infinitesimal (anisotropic) Weyl transformation of the effective action is given by:

$$\delta_\sigma S_{\text{eff}} = \int \sqrt{\gamma} \sigma \mathcal{A} . \quad (3.3)$$

³Such non-relativistic scale anomalies also appear as contact terms in correlation functions of the flat space theory involving the operator D [23].

For Galilean invariant theories in $d + 1$ dimensions, it has been suggested in [15] that these Lifshitz anomalies correspond to conformal anomalies of relativistic field theories in $d + 2$ dimensions defined on a manifold with a null isometry, via a null reduction procedure. In particular, this suggests a possible A-type anomaly in these Galilean theories, which corresponds to the Euler density anomaly term of the $d + 2$ dimensional relativistic theory. In [17], this possibility was confirmed for a $2 + 1$ dimensional Galilean theory via a cohomological analysis of the Wess-Zumino consistency condition (an explicit expression for this A-type anomaly is given in equation (5.13) of [17]). However, it was observed that this A-type anomaly exists only when one assumes the Frobenius condition is violated by the curved spacetime NC structure, i.e. when the 1-form n_μ does not satisfy:

$$n \wedge dn = 0, \quad (3.4)$$

and therefore does not induce a foliation of the spacetime manifold into equal-time slices. When such a foliation structure exists, this A-type anomaly term becomes cohomologically trivial, and can be removed by adding an appropriate local counter-term to the effective action, of the form:

$$L_{\text{c.t.}} = \frac{1}{2} a^\mu \partial_\mu (a^2) + \frac{3}{8} a^4, \quad (3.5)$$

where $a_\mu \equiv -\mathcal{L}_v n_\mu$ is the acceleration associated with n_μ (see equation (5.14) in [17]).

The existence of the A-type scale anomaly in the Galilean case suggests it may be possible to follow a similar argument to the one given in [9] to prove an a-theorem for Galilean field theories. Consider a theory which is invariant under the Galilean group, that flows between a UV and an IR $z = 2$ Lifshitz fixed points. This theory can be coupled to a non-relativistic dilaton field τ by multiplying any dimensionful parameter by an appropriate exponent of τ compensating for its dimension, thereby rendering the theory scale invariant. We can also add a kinetic term for the dilaton, however as we have seen in previous sections in order to have a boost invariant, dynamic term one has to involve the $U(1)$ particle number Goldstone mode θ . We can then choose this kinetic term to be of the form (2.9)⁴ (with an arbitrary value of m). The coupling of the matter to the dilaton can be made arbitrarily weak by taking the coefficient of this term A to be much larger than all other dimensionful parameters.

Similarly to the way equation (3.2) in [9] was derived, we can write an IR effective action for the non-relativistic Galilean theory coupled to the dilaton. As in the relativistic case, the IR effective action will have a contribution NR_{IR} from the non-relativistic Lifshitz invariant Galilean field theory (that replaces the relativistic CFT) in the infrared. It will have a contribution from the invariant terms, L_{dilaton} , corresponding to the local effective action of the dilaton as discussed in the previous sections, which is of the general form (2.26) (with possibly more terms of the 2 time / 4 space derivative order or higher). And finally, it will have a contribution from the A-type anomaly of the theory. This contribution can be calculated by replacing the Weyl parameter σ in (3.3) by τ , substituting into \mathcal{A}

⁴Alternatively we could discuss an RG flow triggered by some operator acquiring a VEV that spontaneously breaks both scale invariance and the $U(1)$ particle number symmetry, leading to a dynamic NGB effective action of the form discussed in previous sections.

the expression for the A-type anomaly (given in [17] for $2 + 1$ dimensions) evaluated on a NC background which is Weyl transformed⁵ with τ as the parameter, solving the resulting equation and restricting to flat space (see e.g. [14]). Alternatively, it can be obtained from the conformal A-type anomaly contribution in $d + 2$ dimensions via a null reduction. The result for $2 + 1$ dimensions is similar to the $3 + 1$ dimensional relativistic case, and given by:

$$S_{\text{IR}} = \text{NR}_{\text{IR}} + \int dt d^2x \left(L_{\text{dilaton}} + (a_{\text{UV}} - a_{\text{IR}}) \left(-4 (\partial_i \tau)^2 \partial_i^2 \tau - 2 (\partial_i \tau)^4 \right) \right), \quad (3.6)$$

where a_{UV} and a_{IR} are the coefficients of the non-relativistic A-type anomaly in the UV and the IR theories respectively (including the dilaton contribution). Note that the A-type anomaly contribution has no time derivatives. This is a consequence of the fact that it is a $U(1)$ singlet.

Similarly to the relativistic case, the dilaton action L_{dilaton} has additional higher order scale invariant terms, whose couplings are non-universal and cannot be fixed. However, there is an important difference: In the relativistic case, the 4 derivatives terms in L_{dilaton} are distinguishable from the contribution of the A-type anomaly, that is the anomaly contribution term cannot be reproduced by a linear combination of the allowed invariant terms in L_{dilaton} . This is not true for the Galilean case in $2 + 1$ dimensions, since the anomaly contribution term in (3.6) can also be obtained from a local contribution of the form (3.5) to L_{dilaton} . This observation can be understood as a consequence of the Frobenius condition: In order to obtain the dilaton effective action in flat space, one naturally works with a conformally flat background, which necessarily satisfies the Frobenius condition (3.4) and therefore has a foliation structure. On such a background, as discussed in [17], there is no A-type anomaly, as it becomes cohomologically trivial. In order to be able to distinguish the anomaly contribution from non-universal contributions to L_{dilaton} , one would have to instead look at a field theory defined on a curved NC background that violates the Frobenius condition.

It is also important to note here the role played by the special conformal transformation. In the relativistic case, the contribution of the A-type anomaly to the dilaton effective action is invariant under global scaling transformations. It is not, however, invariant under special conformal transformations, and it is this property that prevents it from being included in L_{dilaton} from the point of view of the flat space theory. In the non-relativistic case, imposing full Schrödinger invariance does not have the same consequence, as the (purely spatial) contribution of the A-type anomaly is in fact invariant under the Schrödinger special conformal transformations.

Another difference compared to the relativistic case is that these higher order terms in L_{dilaton} may contribute to the dilaton 2 to 2 scattering, in contrast to the relativistic CFT case where the higher order terms didn't contribute at leading order. There are two other notable differences that have already been mentioned: The first is that, while the

⁵Since we are assuming the $U(1)$ symmetry is not anomalous, the anomalous contribution is gauge invariant, and so will not depend on θ if a gauge transformation is performed with θ as the parameter.

non-relativistic anomaly term is a $U(1)$ singlet (3.6), the dilaton effective action L_{dilaton} in the non-relativistic case involves the $U(1)$ particle number Goldstone mode. Second, in the relativistic CFT case where the RG flow is triggered by conformal SSB, a cosmological constant leads to a gapped mode in contradiction to the Goldstone theorem, and is not allowed in the SSB effective action. On the other hand, in the non-relativistic case as we showed above, a cosmological constant is allowed and simply leads to a chemical potential.

These differences, and especially the first one, namely the fact that in a conformally flat background the A-type anomaly contribution is indistinguishable from that of a trivial term, seem to suggest that it is not straightforward to generalize the proof of the a-theorem to the Galilean case.

4 Summary and Outlook

We studied the mechanism of spontaneous symmetry breaking of scale invariance in non-relativistic field theories that possess Galilean boost invariance. We showed that there is a dynamical gapless mode only if the $U(1)$ particle number symmetry is spontaneously broken too. The dispersion relation of the gapless mode depends on the chemical potential: When the chemical potential is nonzero it is linear and it is quadratic otherwise. We constructed the leading terms in the dilaton effective action in two ways: First by using symmetry arguments and second by employing the Newton-Cartan geometrical structure.

The effective action of the dilaton in relativistic field theories encodes the information of trace anomalies. We considered the question whether and how the non-relativistic scale anomalies are encoded in the non-relativistic dilaton effective action. We found that there is a major difference between the relativistic and non-relativistic cases. The construction of the dilaton effective action in flat space requires working with a conformally flat background. However, such a background satisfies the Frobenius condition and therefore implies that the A-type anomaly is cohomologically trivial [17]. Thus, in contrast to the relativistic case, in order to distinguish the anomaly contribution from non-universal contributions to the dilaton action, one has to consider a curved NC background that violates the Frobenius condition. The study of such field theories and their consistency is an important challenge that we leave for the future. This is likely to shed light on the question whether non-relativistic RG theorems analogous to the relativistic ones exist. Another interesting topic left for the future, which is potentially linked to the structure of the RG flows, is the role of the special conformal generator in the symmetry algebra of non-relativistic field theories. This is the non-relativistic version of scale versus conformal invariance of relativistic field theories.

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